Unstationary thermal conduction

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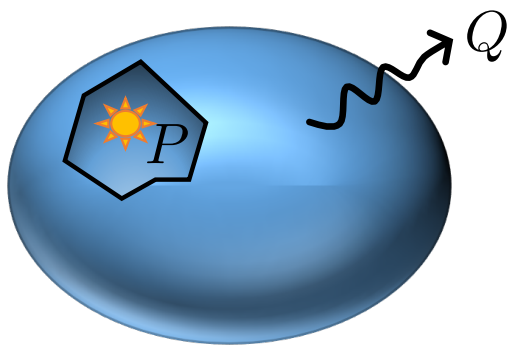
Outline

Energy balance equation

The *thermal energy production* P in a region $\tilde{V}$ during a time interval τ is equal to the sum of the *internal energy storage* Δu in $\tilde{V}$ and the *heat* Q that flows across the boundary of the region $\tilde{V}$ in the same interval τ.

$$P = \Delta u + Q \quad \text{(in joule)}$$

A distinctive feature of all balance laws is that they can be applied to regions of whatever shape and extension and for whatever interval of time.



Energy balance equation: local formulation

By letting the volume $\tilde{V}$ and the time interval τ approach to zero (infinitesimal)

$$\int_\tau \int_{\tilde{V}} \sigma \mathrm{d}V \mathrm{d}t = \Delta u + \int_\tau \oint_{\partial \tilde{V}} \vec{q} \cdot \mathrm{d}\vec{S} \mathrm{d}t$$

σ and $\vec{q}$ integrable

- $\vec{q}$: heat flux density ($\mathrm{W\,m^{-2}}$)
- σ: heat source ($\mathrm{W\,m^{-3}}$)

Applying the divergence theorem

$$\int_\tau \int_{\tilde{V}} \sigma \mathrm{d}V \mathrm{d}t = \Delta u + \int_\tau \int_{\tilde{V}} \nabla \cdot \vec{q} \mathrm{d}V \mathrm{d}t$$

divergence theorem
$\vec{q}$ differentiable

Energy balance equation: local formulation

Exploiting the arbitrariety of V

$$\int_\tau \sigma \mathrm{d}t = \Delta U + \int_\tau \nabla \cdot \vec{q} \mathrm{d}t$$

U differentiable in space

- U: internal energy density (Jm^{-3})

Exploiting the arbitrariety of τ

$$\sigma = \frac{\partial U}{\partial t} + \nabla \cdot \vec{q}$$

U differentiable in time

Internal Energy

The *thermal energy production* P in a region $\tilde{V}$ during a time interval τ is related to the temperature by the state equation

$$u = u_0 + CT \quad \text{(in joule)} \tag{1}$$

- C: thermal capacity $(\mathrm{J\,K^{-1}})$

If the volume approaches zero, the internal energy density becomes

$$U = U_0 + \rho c T \tag{2}$$

- ρ: volumetric mass density $(\mathrm{kg\,m^{-3}})$
- c: specific heat $(\mathrm{J\,kg^{-1}\,K^{-1}})$

Fourier's law

The heat flux Φ crossing a plane surface element $\tilde{S}$ is proportional to the area S, to the temperature difference ΔT between two points which lie on the normal to the plane surface element. Moreover it is inversely proportional to the distance Δx between the two points and it has the direction opposite to that of the temperature increase:

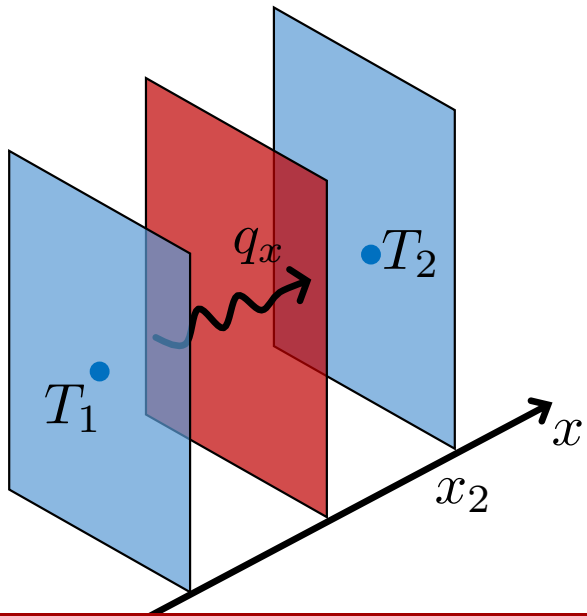
$$\frac{\Phi}{S} = -\lambda \frac{\Delta T}{\Delta x}$$

by letting Δx approach zero:

$$q_x = \frac{\Phi}{S} = -\lambda \frac{\mathrm{d}T}{\mathrm{d}x}$$

or, more generally:

$$\vec{q} = -\lambda \nabla T$$



Differential (local) view

Conservation of energy law:

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{q} = \sigma$$

Fourier's law:

$$\vec{q} = -\lambda \nabla T = -\lambda \vec{g}$$

State equation:

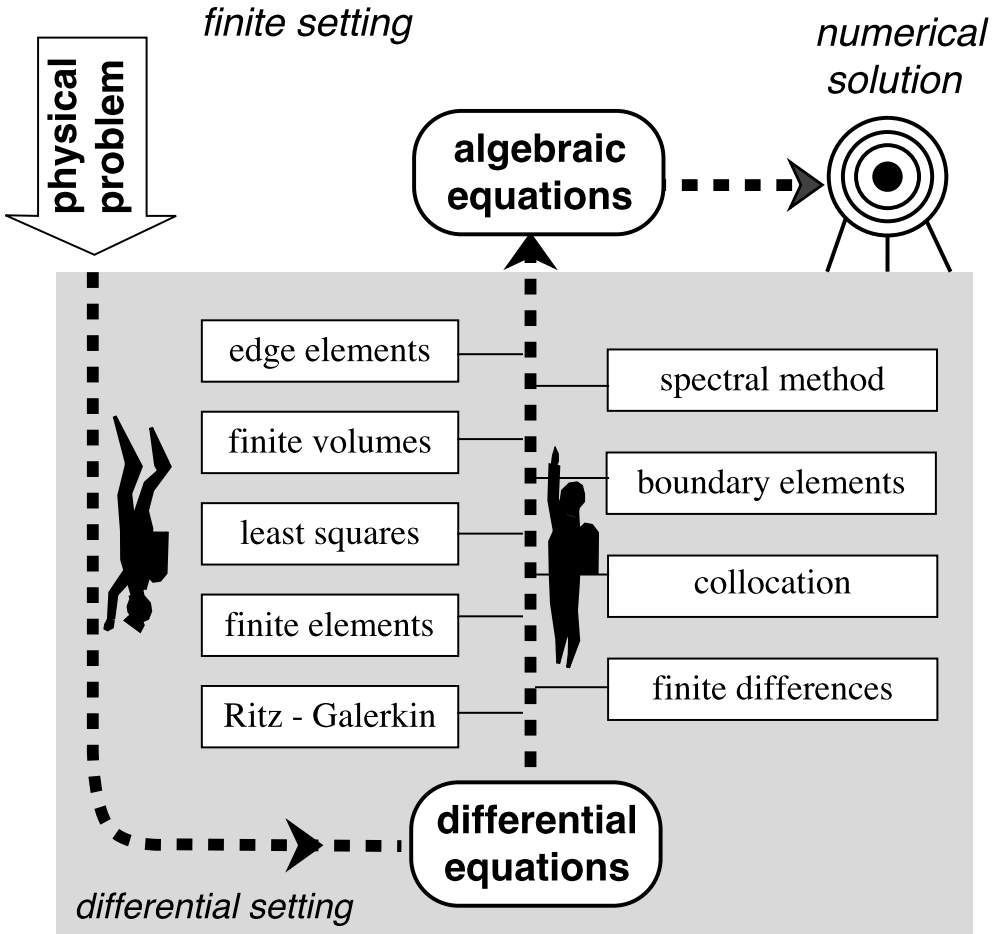
$$U = U_0 + \rho c T$$

By substitution, one obtains the parabolic partial differential equation:

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot \lambda \nabla T = \sigma$$

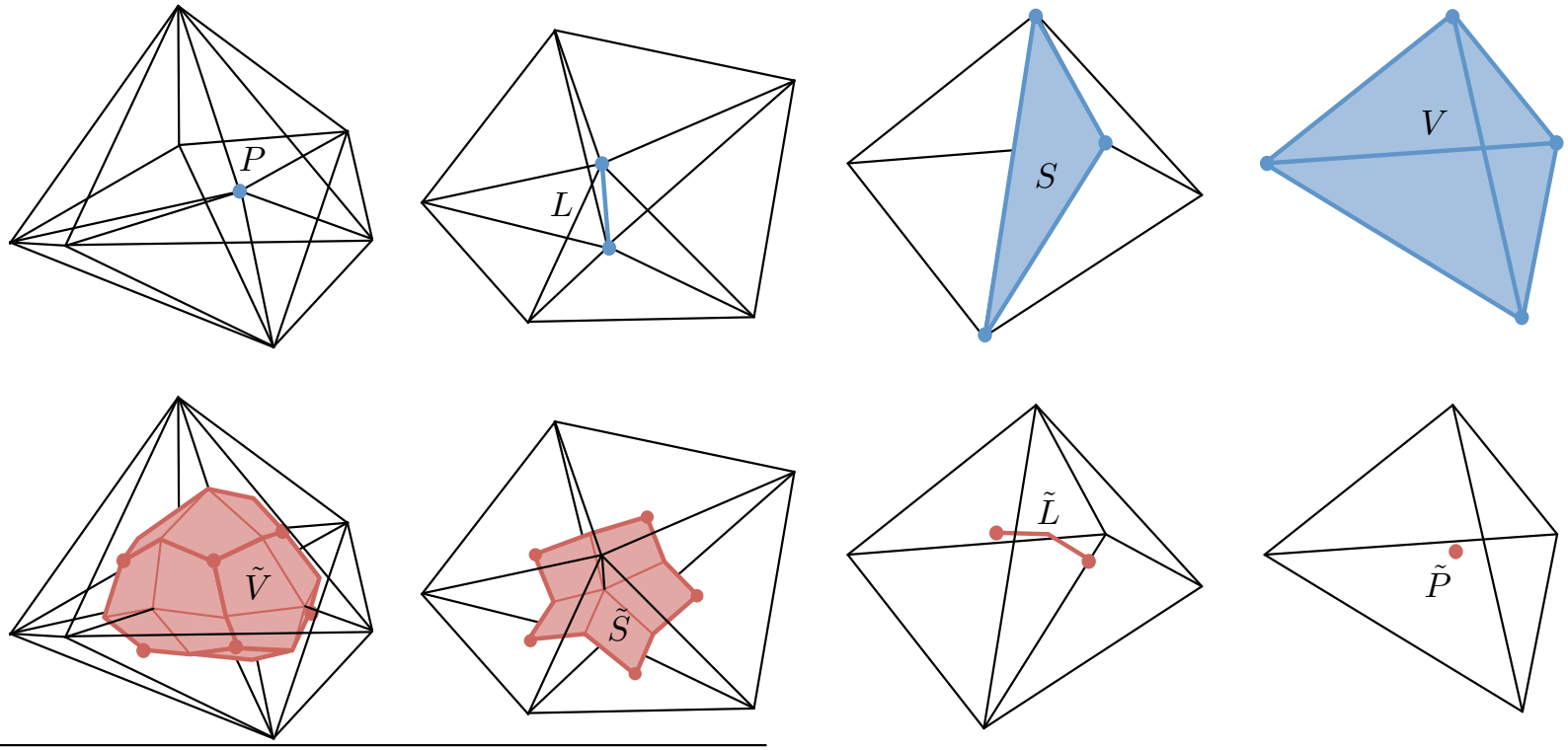
to be discretized...

Differential (local) view



Global variables

An alternative way to directly obtain a discrete system is to resort to *global variables*, i.e. quantities associated to space entities.
Using global variables a straightforward formulation of the problem can be obtained.[1]



Association to spatial elements: source variables

- the heat source σ is associated to the volume $\tilde{V}$ of the region where it is located:

$$p = \int_{\tilde{V}} \sigma \mathrm{d}V$$

Through the energy balance equation, the heat source equals equals the rate of change of the internal energy density U plus $\nabla \cdot \vec{q}$. Also these quantities are associated to dual volumes:

- internal energy:

$$u = \int_{\tilde{V}} U \mathrm{d}V$$

- heat flux:

$$\Phi = \oint_{\tilde{S}=\partial\tilde{V}} \vec{q} \cdot \mathrm{d}\vec{S} = \int_{\tilde{V}} \nabla \cdot \vec{q} \mathrm{d}V$$

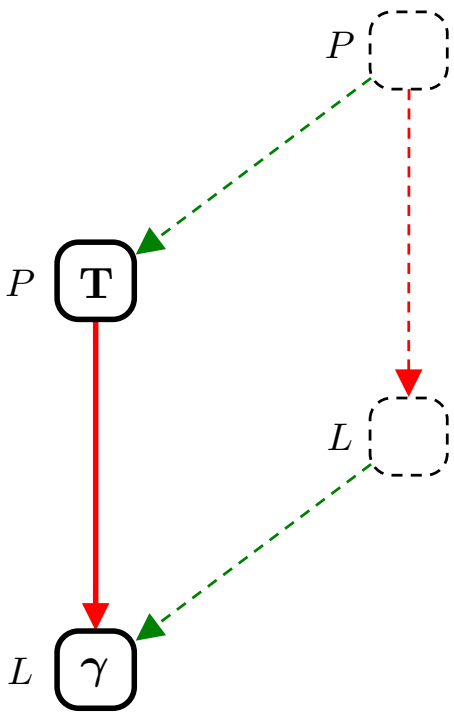
?

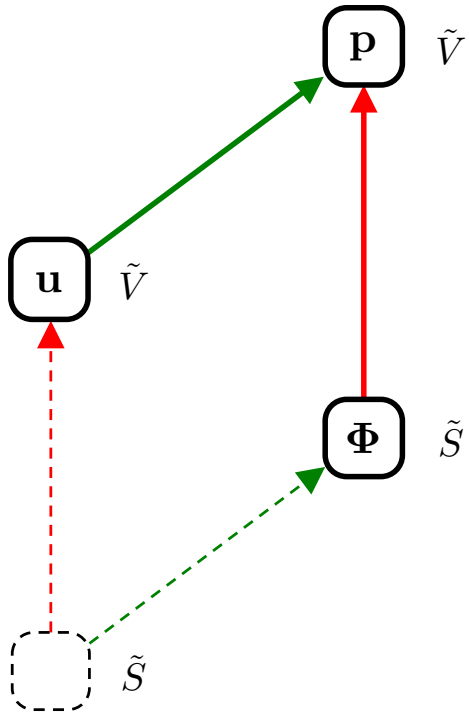
Association to spatial elements: configuration variables

- Temperature T describes the configuration of the system and is associated to points
- As a consequence the temperature difference is associated to the edge connecting the two points

$$\gamma = \int_L \nabla T \cdot \mathrm{d}\vec{l}$$

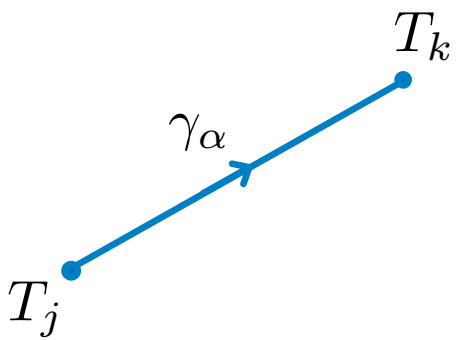
Association to spatial elements: Tonti diagram



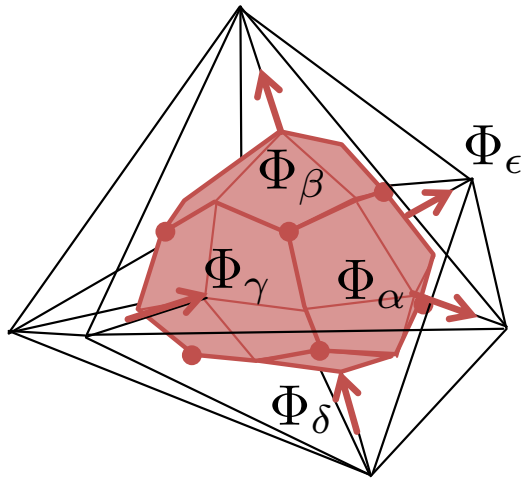


Topological equations

primal edge



dual volume



$$\gamma_\alpha = +T_k - T_j$$

$$\boldsymbol{\gamma} = \mathbf{G}\mathbf{T}$$

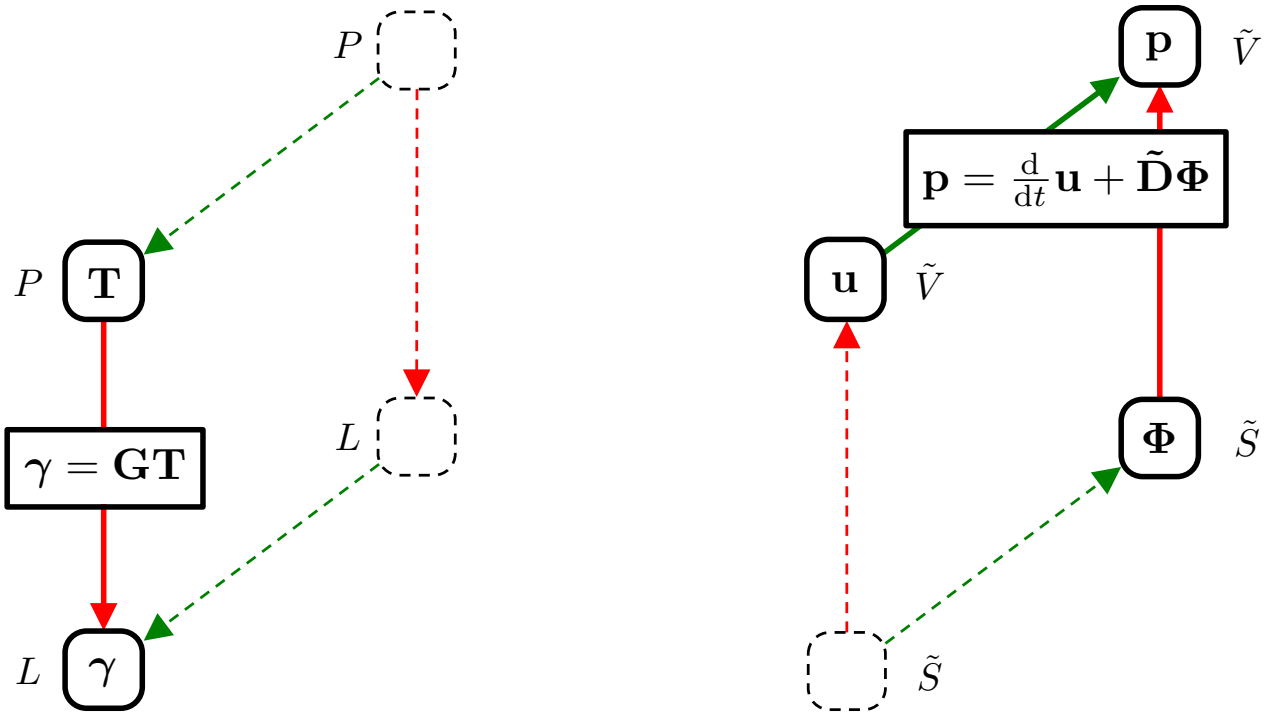
$$\{\mathbf{G}\}_{ij} \in \{-1, 0 + 1\}$$

$$+\Phi_\alpha + \Phi_\beta - \Phi_\gamma - \Phi_\delta + \Phi_\epsilon = p_i - \frac{\mathrm{d}}{\mathrm{d}t} u_i$$

$$\tilde{\mathbf{D}}\boldsymbol{\Phi} = \mathbf{p} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}$$

$$\{\tilde{\mathbf{D}}\}_{ij} \in \{-1, 0 + 1\}$$

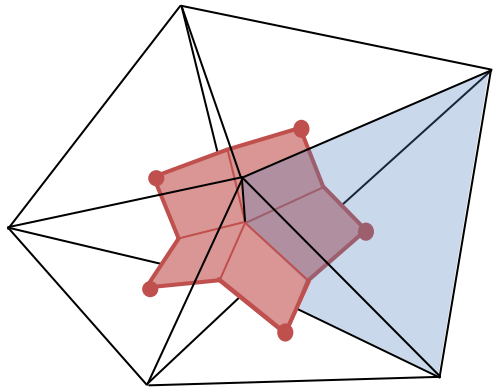
Topological equations: Tonti diagram



Temperature difference-heat flux

By using Whitney elements for interpolating the temperature gradient

$$\nabla T(\vec{r}) = \sum_{i=1}^{6} \vec{w}_i^E(\vec{r})\gamma_i$$



$$\Phi'_k = \int_{\tilde{S}'_k} \vec{q}\cdot \mathrm{d}\vec{S} = \int_{\tilde{S}'_k} -\lambda\nabla T\cdot \mathrm{d}\vec{S} = \int_{\tilde{S}'_k} \sigma\left(\sum_{i=1}^{6}\vec{w}_i^E\gamma_i\right)\cdot \mathrm{d}\vec{S}$$

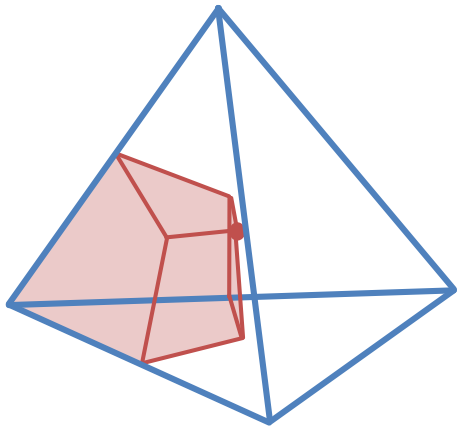
$$= -\sum_{i=1}^{6}\gamma_i \int_{\tilde{S}'_k} \lambda\vec{w}_i^E\cdot \mathrm{d}\vec{S} = -\sum_{i=1}^{6} m'_{\lambda,ki}\gamma_i$$

$$\mathbf{\Phi} = -\mathbf{M}_\lambda\boldsymbol{\gamma}$$

Temperature-internal energy: nodal basis

The internal energy in each portion of the jth dual volume that belongs to a tetrahedron is given by:

$$u'_j = \int_{\tilde{V}_j} U_0 \mathrm{d}V + \int_{\tilde{V}_j} \rho c T \mathrm{d}V = u'_0 + \int_{\tilde{V}_j} \rho c T \mathrm{d}V$$



Temperature is then interpolated by means of nodal shape functions

$$u'_j = u'_0 + \int_{\tilde{V}_j} \rho c \sum_{i=1}^{4} w_i^N T_i \mathrm{d}V = u'_0 + \sum_{i=1}^{4} T_i \int_{\tilde{V}_j} \rho c w_i^N \mathrm{d}V$$

Neglecting the constant term u'_0:

$$m'_{\rho c, ji} = \int_{\tilde{V}_j} \rho c w_i^N \mathrm{d}V$$

Temperature-internal energy: pulse basis

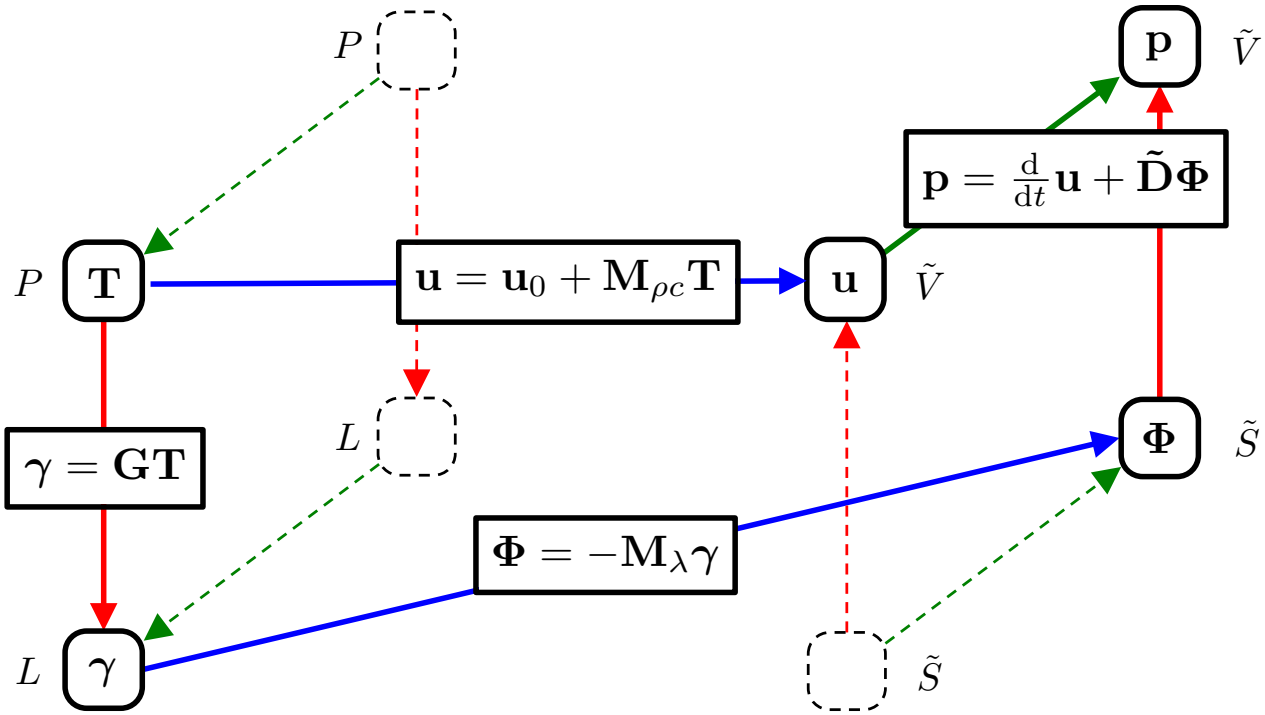
The use of nodal functions gives rise to sparse matrices, with non-null off diagonal elements
Idea: consider the temperature inside each dual volume as constant and equal to the temperature of the corresponding primal node (piece-wise constant interpolating functions):

$$m'_{\rho c,ji} = \int_{\tilde{V}_j} \rho c \, p_i^N \, \mathrm{d}V = \begin{cases} \int_{\tilde{V}_j} \rho c \mathrm{d}V & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{M}_{\rho c}\mathbf{T}$$

- the time integration method can be explicit
- equivalent to mass lumping

Constitutive equations: Tonti diagram



Algebraic (global) view

Conservation of energy law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + \tilde{\mathbf{D}}\mathbf{\Phi} = \mathbf{p}$$

Fourier's law:

$$\mathbf{\Phi} = -\mathbf{M}_\lambda \boldsymbol{\gamma}$$

State equation:

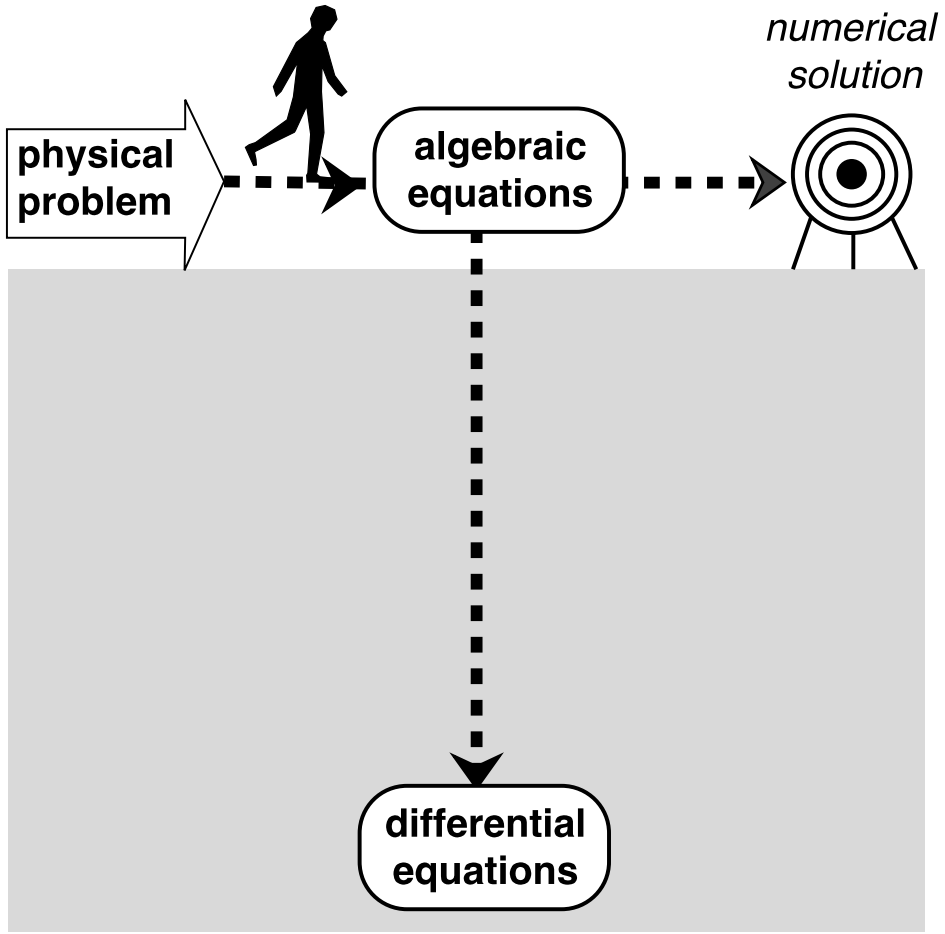
$$\mathbf{u} = \mathbf{u}_0 + \mathbf{M}_{\rho c}\mathbf{T}$$

By substitution, considering that $\tilde{\mathbf{D}} = -\mathbf{G}^{\mathrm{T}}$

$$\mathbf{M}_{\rho c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T} + \mathbf{G}^{\mathrm{T}}\mathbf{M}_\lambda \mathbf{G}\mathbf{T} = \mathbf{p}$$

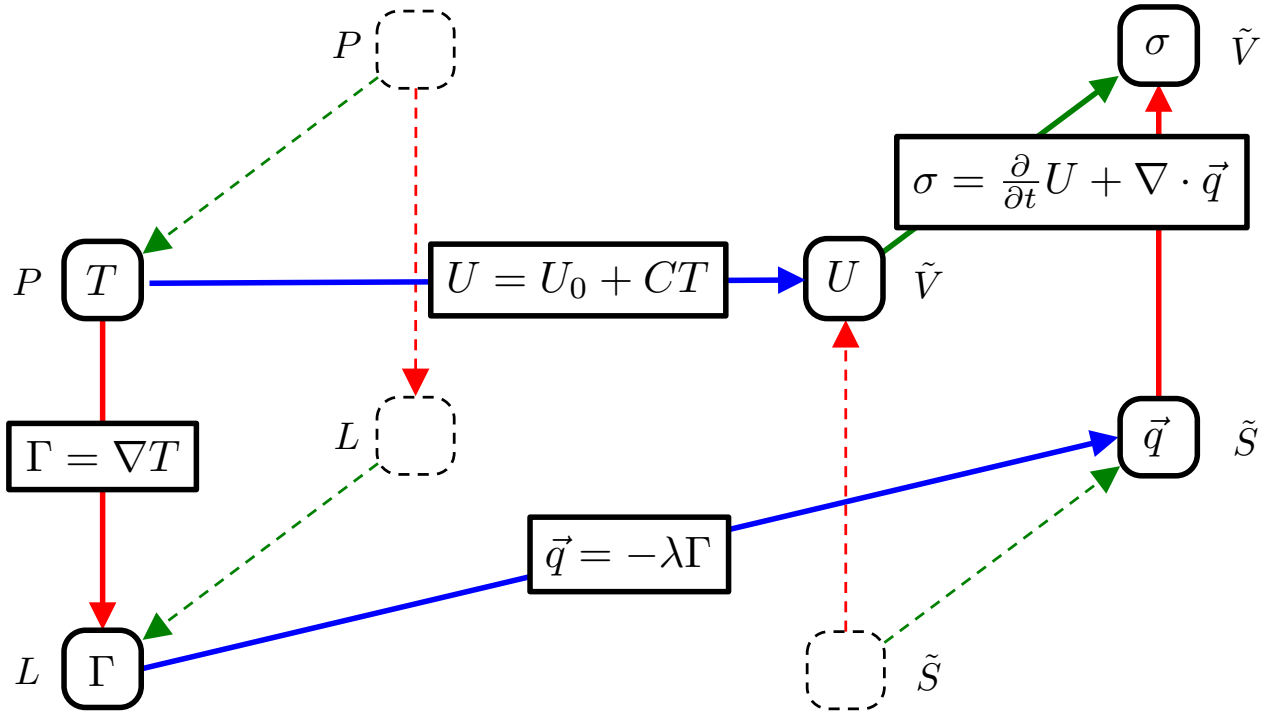
Ready to be solved!

Algebraic (global) view



Cell method: recipe for practitioners

- associate variables to primal/dual cell complex (Tonti diagram)



Cell method: recipe for practitioners

- convert point-wise quantities into global ones by integration
- convert the differential operators into their algebraic counterparts

$$\nabla \rightarrow \mathbf{G}, \tilde{\mathbf{G}}$$
$$\nabla\times \rightarrow \mathbf{C}, \tilde{\mathbf{C}}$$
$$\nabla\cdot \rightarrow \mathbf{D}, \tilde{\mathbf{D}}$$

- build the constitutive matrices
  - primal node - dual volume (e.g. thermal capacitance matrix)
  - primal edge - dual surface (e.g. thermal conductance matrix)
  - primal face - dual edge (e.g. reluctance matrix)
  - primal volume - dual node (???)

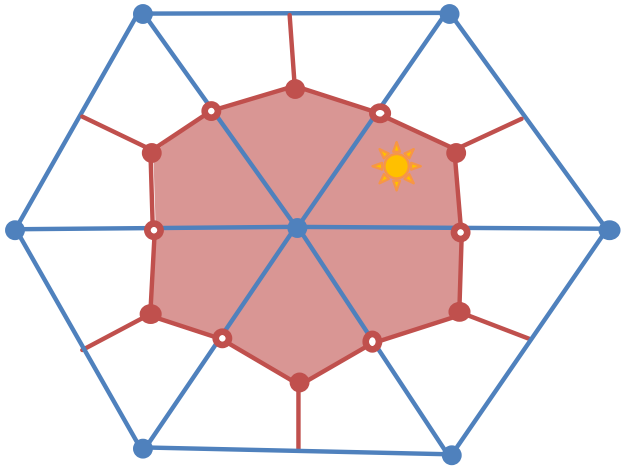
Right hand side formation

If power density σ is assigned to a region, the power density must be integrated with respect to the dual volumes:

$$p_i = \int_{\tilde{V}_i} \sigma \mathrm{d}V \approx \sigma \tilde{V}_i$$

If the source S is concentrated (pointwise), it is assigned to the dual volume it belongs to

$$p_i = S$$



Dirichlet boundary conditions

Dirichlet boundary conditions have the form

$$T = T_{\mathrm{D}} \quad \text{on a portion of the boundary}$$

Decompose the unknown vector in $[\mathbf{T} \quad \mathbf{T}_{\mathrm{D}}]^{\mathrm{T}}$:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{\mathrm{D}} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{T}_{\mathrm{D}} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{p}_{\mathrm{D}} \end{bmatrix}.$$

The free unknowns can be obtained by solving the first block row:

$$\mathbf{A}_{11}\mathbf{T} = \mathbf{p} - \mathbf{A}_{12}\mathbf{T}_{\mathrm{D}}$$

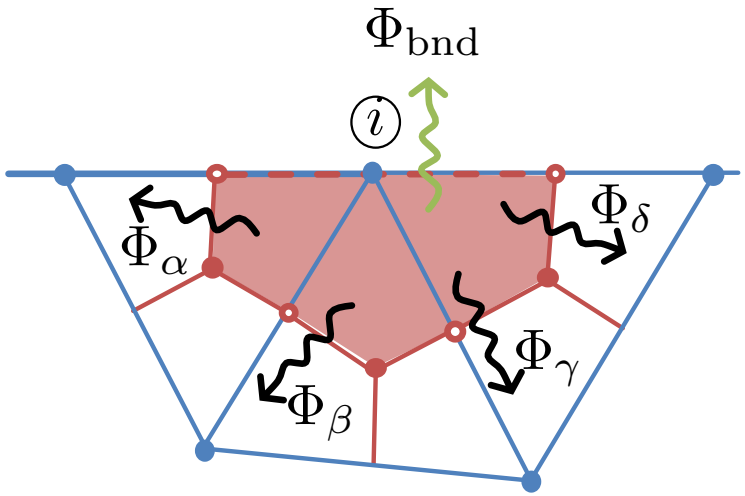
Natural boundary conditions

The energy balance (steady state) on a boundary dual volume reads:

$$\Phi_\alpha + \Phi_\beta + \Phi_\gamma + \Phi_\delta + \Phi_{\text{bnd}} = p_i$$

If Φ_{bnd} is not explicitly imposed, the natural condition is

$$\Phi_{\text{bnd}} = 0 \rightarrow \text{adiabatic}$$

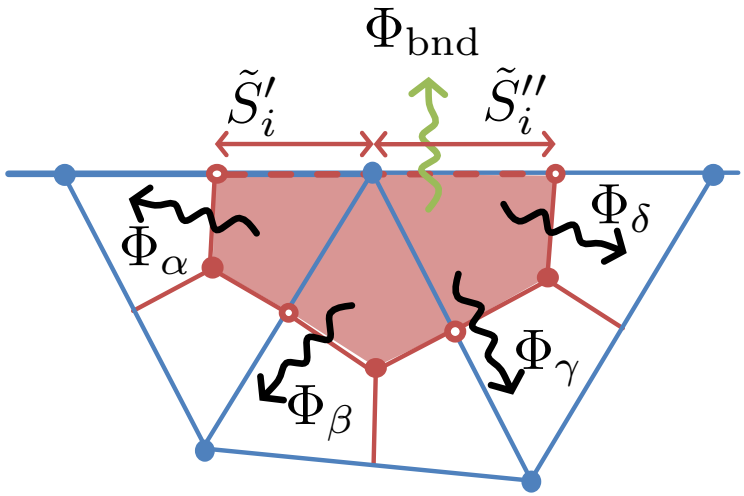


Neumann boundary conditions

If the (outward) flux density q_0 through a portion of the boundary is known, it is possible to assign an additional contribution to the rhs:

$$\Phi_\alpha + \Phi_\beta + \Phi_\gamma + \Phi_\delta = p_i - \Phi_{\mathrm{bnd}}$$

$$\Phi_{\mathrm{bnd}} = q_0(\tilde{S}_i' + \tilde{S}_i'')$$



Robin boundary conditions

Convection boundary conditions are a special case of Robin boundary conditions

$$au + b\frac{\partial u}{\partial n} = g \quad \text{Robin b.c.}$$

In fact

$$q_0 = h(T - T_0) \rightarrow hT - q_0 = hT_0$$

$$a = h; \quad b = -1; \quad g = hT_0$$

In terms of global variables

$$\Phi_{\text{bnd}} = h(\tilde{S}_i' + \tilde{S}_i'')(T - T_0)$$

Robin boundary conditions

In matrix form

$$\tilde{\mathbf{D}}\mathbf{\Phi} = -h\tilde{\mathbf{S}}(\mathbf{T} - \mathbf{T}_0)$$

$h\tilde{\mathbf{S}}$ is a diagonal matrix that contributes to both the stiffness matrix and the rhs

$$(\mathbf{G}^{\mathrm{T}}\mathbf{M}_\lambda\mathbf{G} + h\tilde{\mathbf{S}})\mathbf{T} = \mathbf{p} + h\tilde{\mathbf{S}}\mathbf{T}_0$$

Exercise

Challenge: Formulate the magnetostatics problem with the cell method in terms of magnetic vector potential:

- solenoidality of the magnetic flux density

$$\nabla \cdot \vec{B} = 0 \rightarrow \vec{B} = \nabla \times \vec{A}$$

- Ampere's law

$$\nabla \times \vec{H} = \vec{J}$$

- constitutive equation

$$\vec{H} = \nu \vec{B}$$

Finally

$$\nabla \times (\nu \nabla \times \vec{A}) = \vec{J}$$