Elastic Problem

Chapter 1: Identification of the variables

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January 2016

Outline

Displacement

Definition

- Considering an object in the space, each point of its volume has a defined position with respect to the reference
- when a force acts, the position of each point can change, due to two effects:
  - rigid body motion, i.e. rototranslation in the space
  - deformation of the body shape
- displacement is defined as the difference in position of the same point before and after force action

$$u = \begin{bmatrix} x' - x \\ y' - y \\ z' - z \end{bmatrix} \quad (1)$$

Rigid Displacement

Definition

- Two kinds of rigid motion can be highlighted:
  - rigid translation

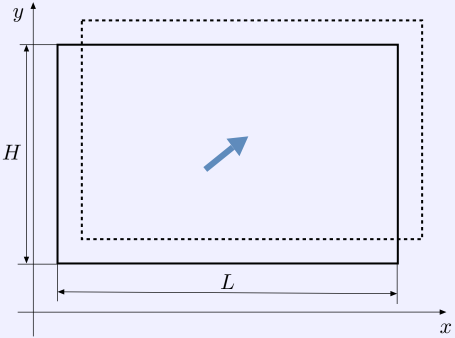


Figure: rigid translation

Rigid Displacement

Definition

- Two kinds of rigid motion can be highlighted:
  - rigid rotation

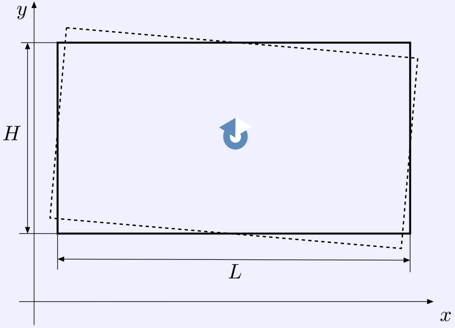


Figure: rigid rotation

Definition

- Two kinds of deformation can be highlighted:
  - dilatation

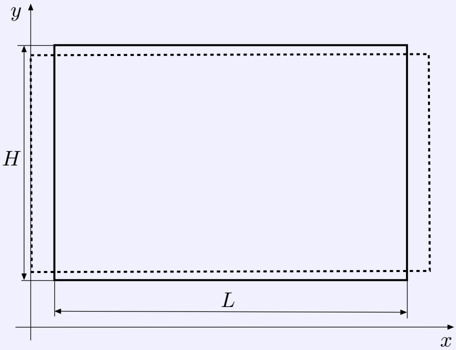


Figure: dilatation

Deformation

Definition

- Two kinds of deformation can be highlighted:
  - distortion

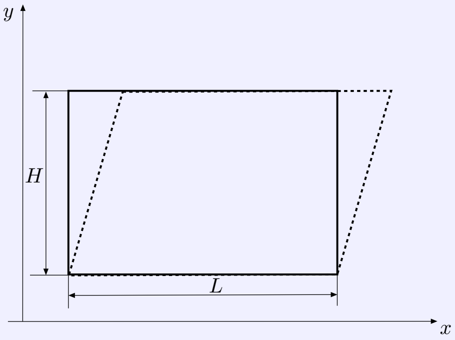


Figure: distortion

Jacobian matrix

- considering all the previous effects, point P moves to P' and in the same way Q moves in Q'. The motion can be described by two vectors dU_P and dU_Q. So the initial segment U changes in U' and vectorially speaking

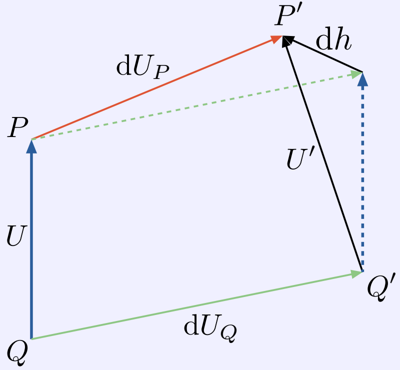


Figure: vector

Jacobian matrix

- due to the vectorial properties, the changed configuration can be expressed as

$$dU_P + U = dU_Q + U' = dU_Q + U + dh \tag{2}$$

- simplifying...

$$dh = dU_P - dU_Q \tag{3}$$

  that means the rigid translation is not relevant for body shape modification that is due to rigid rotation and deformation

- in matricial form:

$$dh = \begin{bmatrix} dx_P - dx_Q \\ dy_P - dy_Q \\ dz_P - dz_Q \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} dU = JdU \tag{4}$$

Jacobian matrix

- Jacobian matrix can be decomposed...

$$[\mathbf{J}] = \frac{1}{2}[\mathbf{J}] - \frac{1}{2}[\mathbf{J}]^T + \frac{1}{2}[\mathbf{J}] + \frac{1}{2}[\mathbf{J}]^T \qquad (5)$$

where the first two adding terms give the rotation matrix

$$[\Omega] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{bmatrix} \qquad (6)$$

Jacobian matrix

- whereas the second ones give the deformation matrix

$$[\varepsilon] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} = \tag{7}$$

$$\begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{8}$$

Jacobian matrix

- in vectorial form

$$\varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [\partial]\, u \qquad (9)$$

Matrix terms physical meaning

- the diagonal terms (ε) means "dilatation"

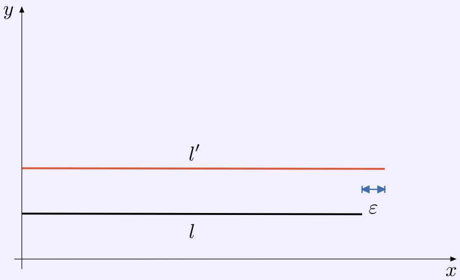


Figure: dilatation

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \cong \frac{l' - l}{l} = \frac{u}{l} \quad (10)$$

Matrix terms physical meaning

- the other terms (γ) means "distorsion"

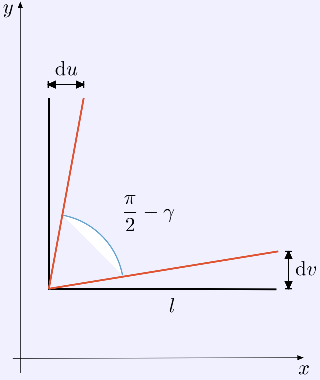


Figure: distorsion

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{11}$$

Surface and Volume forces

- Considering an infinitesimal volume, the equilibrium can be written

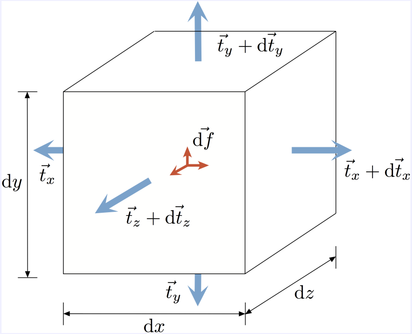


Figure: Surface and Volume forces in an infinitesimal volume, in cyan the normal surface forces, and in red the volume forces

Surface and Volume forces

- all the surface contributions minus the volume force

$$\vec{d}t_x + \vec{d}t_y + \vec{d}t_z - \vec{d}f = 0 \quad (12)$$

- That means the surface forces difference across the volume is equal to zero if no volume forces are present. In fact the force variation across the volume is equal to the volume force contribution, as a consequence

$$\vec{d}t = \begin{bmatrix} dt_x \\ dt_y \\ dt_z \end{bmatrix} = \begin{bmatrix} df_x \\ df_y \\ df_z \end{bmatrix} \quad (13)$$

Stress definition

- Stress is defined as

$$\lim_{S \longrightarrow 0} \frac{dt}{dS} = \sigma \tag{14}$$

- and due to the infinitesimal dimension of the area, moment is

$$\lim_{S \longrightarrow 0} \frac{dm}{dS} = 0 \tag{15}$$

Stress definition

- stress can be divided into two types: normal and tangential. The first is normal to the surface where it acts and it is usually called σ, the second is applied on the surface and acts along one of the other two directions and it is usually called τ.

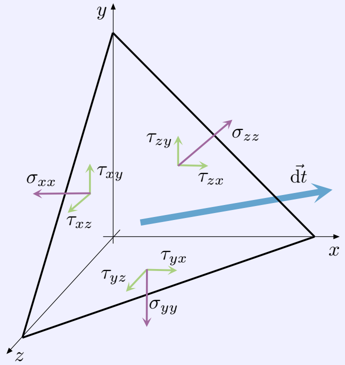


Figure: Normal (red arrows) and shear (green arrows) stress

Stress definition

- considering expression 14 and figure 9, the force increment can be rewritten as

$$\vec{dt} = \begin{bmatrix} \sigma_x \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix} dydz + \begin{bmatrix} \tau_{yx} \\ \sigma_y \\ \tau_{yz} \end{bmatrix} dxdz + \begin{bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_z \end{bmatrix} dxdy \qquad (16)$$

- equation 12 can be exploded along Cartesian directions, writing (for example along x)

$$(\frac{\partial \sigma_x}{\partial x} dx)dydz + (\frac{\partial \tau_{xy}}{\partial y} dy)dxdz + (\frac{\partial \tau_{xz}}{\partial z} dz)dydx - df_x dxdydz = 0 \qquad (17)$$

Equilibrium equation

- Then, by reordering the equation 17 and the other along y and z, in vectorial form it is possible to write:

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} dt_x \\ dt_y \\ dt_z \end{bmatrix} = [\partial]^T dt \qquad (18)$$

Duality

- By observing equation 9 and equation 18, it is possible to note that the differential operator is the same, just transposed. This is the first important element for the analysis of elastic field.
- The CM highlights this duality considering the geometrical quantities where displacements and forces are defined. In fact, displacements are defined related to a point (eq. 1). The dual geometrical element of a point is a volume, and field forces are defined related to a volume (eq. 12)
- Difference of displacements is defined according to an edge (the connection between two nodes, see eq. 4) whereas the surface forces are related to an area (eq. 14). In CM the edge has an area as dual geometrical entities.

Duality

We need "sites" where quantities can be defined

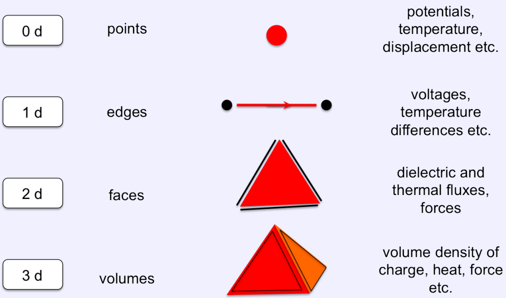


Figure: Geometrical quantities

- Considering the geometrical relationships between elements, it is possible to highlight:

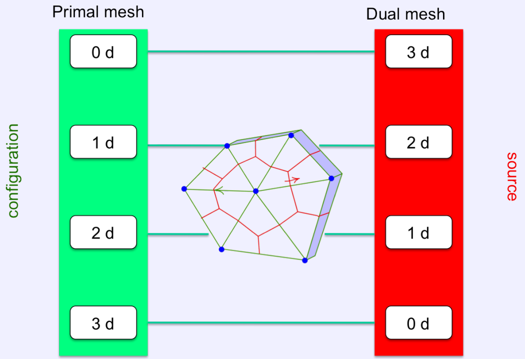
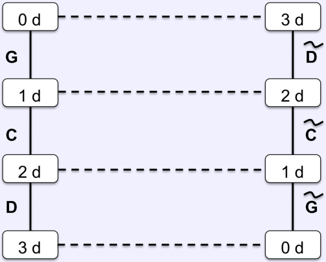


Figure: Geometrical relationship: duality

Duality

- The CM operators are integer, but they are deeply linked to the differential operators and the relationships between differential and integer operators are the same

Vertical links: topological equations (exact)
Horizontal links: constitutive equations (interpolated)



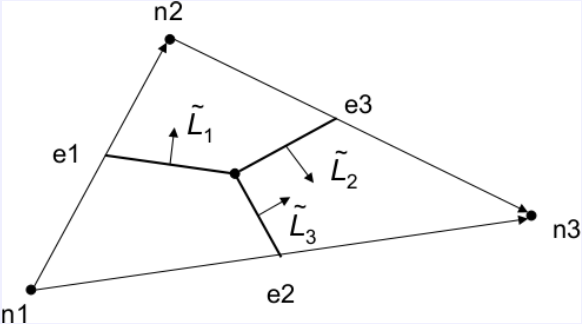
$$\tilde{\mathbf{D}} = -\mathbf{G}^T, \tilde{\mathbf{C}} = \mathbf{C}^T, \tilde{\mathbf{G}} = -\mathbf{D}^T$$

Figure: Operator relationship: duality

Topological equation: 2D example

Discretization

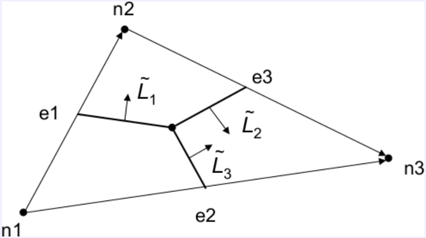
Considering the plane triangle of the figure, it is possible to identify the three nodes n1, n2, n3, the edges that connect the nodes e1, e2, and e3. Each edge is related to a dual surface, in the figure indicated by $\tilde{L}$. The triangle is plane and it has a constant thickness, so the dual surface $\tilde{a}$ is the product of the length $\tilde{L}$ for the thickness δ. The dual volumes are defined by the dual surfaces and they are related to each node.



Configuration equation

Eq. 4 can be rewritten in CM considering an edge and the difference between the displacements of the two related nodes. Consider a plane element made by tree nodes.

$$\begin{Bmatrix} du_{e1} \\ dv_{e1} \\ du_{e2} \\ dv_{e2} \\ du_{e3} \\ dv_{e3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_{n1} \\ v_{n1} \\ u_{n2} \\ v_{n2} \\ u_{n3} \\ v_{n3} \end{Bmatrix} \qquad (19)$$

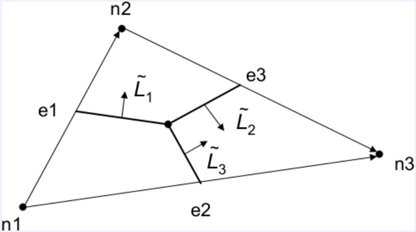


Source equation in CM

Source equation

Eq. 12 can be rewritten in CM considering the dual surface related to an edge ($\tilde{L}$). Consider a plane element made by tree nodes, and positive the surface force t that has outer orientation

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} t_{x\tilde{L}_1} \\ t_{y\tilde{L}_1} \\ t_{x\tilde{L}_2} \\ t_{y\tilde{L}_2} \\ t_{x\tilde{L}_3} \\ t_{y\tilde{L}_3} \end{Bmatrix} = \{f\} \qquad (20)$$



Topological equation

- Referring to the duality expressed before, the eqs. 19 and 20 can be simplified, writing:

$$h = \mathbf{G}u \tag{21}$$

$$-\tilde{\mathbf{D}}t = f \tag{22}$$

- and then, using the duality $\tilde{\mathbf{D}} = -\mathbf{G}^t$ source equilibrium becomes:

$$\mathbf{G}^T t = f \tag{23}$$

... between configuration and source spaces

- What said before is independent on the material of the studied body. The relationship between the source and the configuration spaces has to be related to the material.
- In the linear elastic field, relationship between stress and strain is defined by the material, and it is measured by means of tensile test. In formula

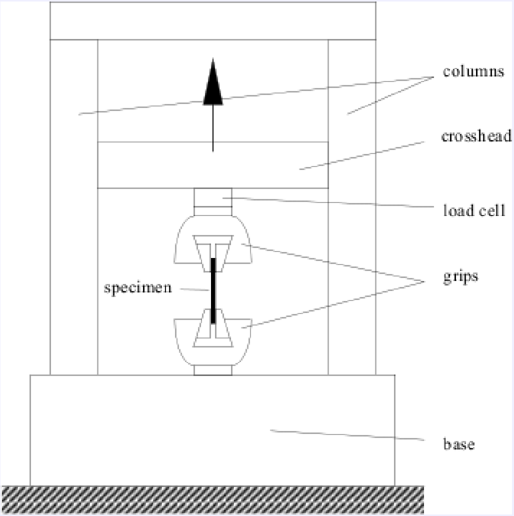
$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \frac{E}{(1-\nu^2)} & \frac{\nu E}{(1-\nu^2)} & 0 \\ \frac{\nu E}{(1-\nu^2)} & \frac{E}{(1-\nu^2)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{array} \right] \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} \qquad (24)$$

where E is the Young's modulus of the material and ν is the Poisson ratio of the material, determined by means of tensile tests.

Constitutive equation

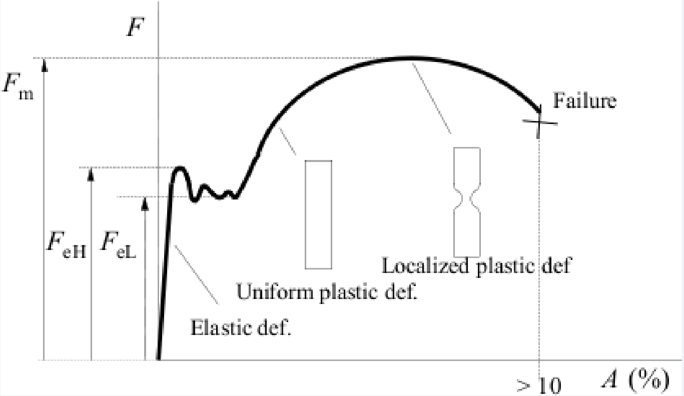
... between configuration and source spaces

- In order to measure the Young modulus a tensile or compression test is performed using a testing machine.



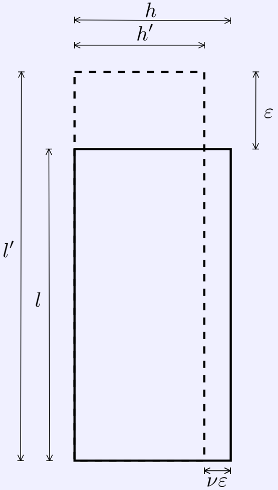
... between configuration and source spaces

- The slope of the linear part of the curve is the Young modulus



... between configuration and source spaces

- Poisson coefficient is the shrinkage of the material measured in the orthogonal direction of tensile stress



... between configuration and source spaces

- in formula, the expression of the transverse strain is:

$$\varepsilon_t = \frac{h' - h}{h} \tag{25}$$

- the relationship between transverse and longitudinal strain is the Poisson ratio, defined as

$$\nu = -\frac{\varepsilon_t}{\varepsilon_l} \tag{26}$$

- The procedure to obtain ν is quite the same used for E, in fact, usually, when a strain gage is used for measure the longitudinal strain, in the same time, another strain gage measures the transverse strain

... between configuration and source spaces

- But the source and the configuration equilibria are not written with respect to respectively stress and strain.
- As first step, stress has to be related to the force, this is possible considering the discretization and the dual surface, consider the three nodes plane element ...

$$\begin{aligned} t_x &= \sigma_x \tilde{a}_x + \tau_{xy} \tilde{a}_y \\ t_y &= \tau_{yx} \tilde{a}_x + \sigma_y \tilde{a}_y \end{aligned} \tag{27}$$

... between configuration and source spaces

- and, writing (27) for each dual face inside a triangle, the relationship between surface force and stress can be written as:

$$\begin{bmatrix} t_{x\tilde{L}_1} \\ t_{y\tilde{L}_1} \\ t_{x\tilde{L}_2} \\ t_{y\tilde{L}_2} \\ t_{x\tilde{L}_3} \\ t_{y\tilde{L}_3} \end{bmatrix} = \delta \begin{bmatrix} \tilde{L}_{1x} & 0 & \tilde{L}_{1y} \\ 0 & \tilde{L}_{1y} & \tilde{L}_{1x} \\ \tilde{L}_{2x} & 0 & \tilde{L}_{2y} \\ 0 & \tilde{L}_{2y} & \tilde{L}_{2x} \\ \tilde{L}_{3x} & 0 & \tilde{L}_{3y} \\ 0 & \tilde{L}_{3y} & \tilde{L}_{3x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{28}$$

where δ is the problem depth. In compact form:

$$t = \mathbf{A}\sigma \tag{29}$$

... between configuration and source spaces

- As second step, strain has to be related to the displacement gradient, this is possible considering the discretization and the primal edge, consider the three nodes plane element ...

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix} = \begin{bmatrix} P_{x1} & 0 & P_{x2} & 0 & 0 & 0 \\ 0 & P_{y1} & 0 & P_{y2} & 0 & 0 \\ P_{y1} & P_{x1} & P_{y2} & P_{x2} & 0 & 0 \end{bmatrix} \begin{bmatrix} du_{e1} \\ dv_{e1} \\ du_{e2} \\ dv_{e2} \\ du_{e3} \\ dv_{e3} \end{bmatrix} \tag{30}$$

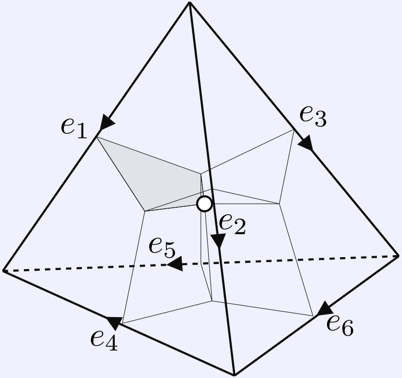
and in compact form

$$\varepsilon = \mathbf{P}h \tag{31}$$

Topological equation: 3D example

Discretization

Considering the tetrahedral element of the figure, it is possible to identify the four nodes n1, n2, n3, n4, the six edges that connect the nodes e1, e2, e3, e4 ,e5, and e6. Each edge is related to a dual surface.The dual volumes are defined by the dual surfaces and they are related to each node.



Configuration equation in CM

Configuration equation for 3D element

Eq. 4 can be rewritten in CM considering an edge and the difference between the displacements of the two related nodes. Consider a 3D tetrahedral element made by four nodes.

$$\begin{bmatrix} h_{x1} \\ h_{y1} \\ h_{z1} \\ h_{x2} \\ h_{y2} \\ h_{z2} \\ h_{x3} \\ h_{y3} \\ h_{z3} \\ h_{x4} \\ h_{y4} \\ h_{z4} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & -\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix} \tag{32}$$

Source equation

Eq. 12 can be rewritten in CM considering the dual surface related to an edge ($\tilde{L}$). Consider a tetrahedral element made by four nodes, and positive the surface force t that has outer orientation

$$\begin{bmatrix} -\mathbf{I}_3 & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & -\mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} t_{x1} \\ t_{y1} \\ t_{z1} \\ t_{x2} \\ t_{y2} \\ t_{z2} \\ t_{x3} \\ t_{y3} \\ t_{z3} \\ t_{x4} \\ t_{y4} \\ t_{z4} \end{bmatrix} = f \qquad (33)$$

Topological equation

- Referring to the duality expressed before, the eqs. 19 and 20 can be simplified, writing:

$$h = \mathbf{G}_3 u \tag{34}$$

$$-\tilde{\mathbf{D}}_3 t = f \tag{35}$$

- and then, using the duality $\tilde{\mathbf{D}}_3 = -\mathbf{G}_3^T$ source equilibrium becomes:

$$\mathbf{G}_3^T t = f \tag{36}$$

... between configuration and source spaces

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} \left[\frac{(1-\nu)E}{(1-2\nu)(1+\nu)}\right]_{3\times3} & \mathbf{0} \\ \mathbf{0} & \left[\frac{E}{(1+\nu)}\right]_{3\times3} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{37}$$

where E is the Young's modulus of the material and ν is the Poisson ratio of the material, determined by means of tensile tests.

Constitutive equation

... between configuration and source spaces

$$\begin{bmatrix} t_{1x} \\ t_{1y} \\ t_{1z} \\ t_{2x} \\ t_{2y} \\ t_{2z} \\ \dots \\ t_{6x} \\ t_{6y} \\ t_{6z} \end{bmatrix} = \begin{bmatrix} \tilde{a}_{x1} & 0 & 0 & \tilde{a}_{y1} & 0 & \tilde{a}_{z1} \\ 0 & \tilde{a}_{y1} & 0 & \tilde{a}_{x1} & \tilde{a}_{z1} & 0 \\ 0 & 0 & \tilde{a}_{z1} & 0 & \tilde{a}_{y1} & \tilde{a}_{x1} \\ \tilde{a}_{x2} & 0 & 0 & \tilde{a}_{y2} & 0 & \tilde{a}_{z2} \\ 0 & \tilde{a}_{y2} & 0 & \tilde{a}_{x2} & \tilde{a}_{z2} & 0 \\ 0 & 0 & \tilde{a}_{z2} & 0 & \tilde{a}_{y2} & \tilde{a}_{x2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{x6} & 0 & 0 & \tilde{a}_{y6} & 0 & \tilde{a}_{z6} \\ 0 & \tilde{a}_{y6} & 0 & \tilde{a}_{x6} & \tilde{a}_{z6} & 0 \\ 0 & 0 & \tilde{a}_{z6} & 0 & \tilde{a}_{y6} & \tilde{a}_{x6} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \quad (38)$$

where $\tilde{a}$ are components of the area vectors of the portion of dual face contained inside a volume. In compact form:

$$\mathbf{t} = \mathbf{A}\sigma \quad (39)$$

... between configuration and source spaces

- As a second step, strain has to be related to the displacement gradient. By considering a tetrahedral element with four nodes and affine behaviour of the displacement components u, v, w with respect to the spatial coordinates, the displacement can be written as a function of (x, y, z) as:

$$u(x, y, z) = H_{xx}x + H_{xy}y + H_{xz}z + c_u \quad (40)$$

$$v(x, y, z) = H_{yx}x + H_{yy}y + H_{yz}z + c_v \quad (41)$$

$$w(x, y, z) = H_{zx}x + H_{yz}y + H_{zz}z + c_w \quad (42)$$

Constitutive equation

... between configuration and source spaces

where the H components are constant. Writing the relative displacements along x direction of the primal nodes one obtains:

$$h_{x1} = u_1 - u_2 = H_{xx}(x_1 - x_2) + H_{xy}(y_1 - y_2) + H_{xz}(z_1 - z_2) \quad (43)$$

$$h_{x2} = u_1 - u_3 = H_{xx}(x_1 - x_3) + H_{xy}(y_1 - y_3) + H_{xz}(z_1 - z_3) \quad (44)$$

$$h_{x3} = u_1 - u_4 = H_{xx}(x_1 - x_4) + H_{xy}(y_1 - y_4) + H_{xz}(z_1 - z_4) \quad (45)$$

$$h_{x4} = u_2 - u_3 = -h_{x1} + h_{x2} \quad (46)$$

$$h_{x5} = u_2 - u_4 = -h_{x1} + h_{x3} \quad (47)$$

$$h_{x6} = u_3 - u_4 = -h_{x2} + h_{x3} \quad (48)$$

$$(49)$$

... between configuration and source spaces

and noting that the last three equations are linearly dependent on the first three, the formulation can be rewritten as:

$$\begin{bmatrix} h_{x1} \\ h_{x2} \\ h_{x3} \end{bmatrix} = \begin{bmatrix} \tilde{L}_{1x} & \tilde{L}_{1y} & \tilde{L}_{1z} \\ \tilde{L}_{2x} & \tilde{L}_{2y} & \tilde{L}_{2z} \\ \tilde{L}_{3x} & \tilde{L}_{3y} & \tilde{L}_{3z} \end{bmatrix} \begin{bmatrix} H_{xx} \\ H_{xy} \\ H_{xz} \end{bmatrix} \tag{50}$$

... between configuration and source spaces

The matrix containing components of the edge vectors are defined as:

$$\mathbf{L} = \begin{bmatrix} \tilde{L}_{1x} & \tilde{L}_{1y} & \tilde{L}_{1z} \\ \tilde{L}_{2x} & \tilde{L}_{2y} & \tilde{L}_{2z} \\ \tilde{L}_{3x} & \tilde{L}_{3y} & \tilde{L}_{3z} \end{bmatrix} \quad (51)$$

In order to compute components H, matrix $\mathbf{P}'$ can be obtained as:

$$\mathbf{P}' = \mathbf{L}^{-1} = \begin{bmatrix} P_{x1} & P_{x2} & P_{x3} \\ P_{y1} & P_{y2} & P_{y3} \\ P_{z1} & P_{z2} & P_{z3} \end{bmatrix} \quad (52)$$

Constitutive equation

... between configuration and source spaces

By operating on the matrices written for each spatial direction, and using the components of matrix $\mathbf{P}'$, one obtains:

$$\mathbf{H}^T = \begin{bmatrix} H_{xx} & H_{yx} & H_{zx} \\ H_{xy} & H_{yy} & H_{zy} \\ H_{xz} & H_{yz} & H_{zz} \end{bmatrix} \tag{53}$$

$$\mathbf{H}^T = \begin{bmatrix} P_{x1} & P_{x2} & P_{x3} \\ P_{y1} & P_{y2} & P_{y3} \\ P_{z1} & P_{z2} & P_{z3} \end{bmatrix} \begin{bmatrix} h_{x1} & h_{y1} & h_{z1} \\ h_{x2} & h_{y2} & h_{z2} \\ h_{x3} & h_{y3} & h_{z3} \end{bmatrix} \tag{54}$$

... between configuration and source spaces

- Now it is possible to write the solution equation, considering the Tonti's diagram of the elastic problem

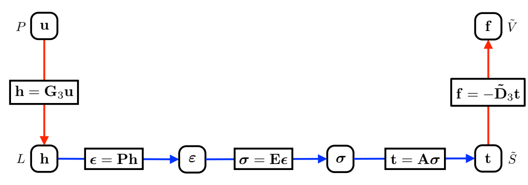


Figure: Tonti's diagram of elastic problem

... between configuration and source spaces

- Then solution equation has the expression

$$\mathbf{G}^T\mathbf{AEPG}u = F_e, \tag{55}$$

and in compact form:

$$\mathbf{G}^T\mathbf{M}_{el}\mathbf{G}u = F_e, \tag{56}$$

- considering the stiffness formulation of the elastic problem, equation 56 can be expressed as:

$$\mathbf{K}u = F_e \tag{57}$$